

Wronskians and Linear Dependence of Formal Power Series

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Abstract

We give a new proof of the fact that the vanishing of generalized Wronskians implies linear dependence of formal power series in several variables. Our results are also valid for quotients of germs of analytic functions.

1 Introduction

Suppose f_1, \dots, f_n are $(n-1)$ -times differentiable functions. If they are linearly dependent over the constants then their Wronskian vanishes, i.e.

$$\det \begin{pmatrix} f_1 & \cdots & f_n \\ f_1' & \cdots & f_n' \\ \vdots & & \vdots \\ f_1^{(n-1)} & \cdots & f_n^{(n-1)} \end{pmatrix} = 0. \quad (*)$$

The converse of this classical fact, however, is not true, even for C^∞ -functions. An example illustrating this was given over a century ago by Bôcher [1]: the functions

$$f_1 = \begin{cases} e^{-1/x^2} & x \neq 0; \\ 0 & x = 0 \end{cases}, \quad \text{and} \quad f_2 = \begin{cases} e^{-1/x^2} & x > 0; \\ 0 & x = 0 \\ 2e^{-1/x^2} & x < 0 \end{cases}$$

are linear independent over \mathbb{R} , yet their Wronskian vanishes identically. Over the years, a number of variations of the converse of $(*)$ had been considered [2, 8, 13, 14]. In the several variables case, with Wronskian replaced by generalized

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Wronskians, the converse of (*) was proved for analytic functions. Bostan and Dumas gave a short proof of this result in [2]. They proved, more generally, that if K is a field of characteristic zero and formal power series $f_1, \dots, f_n \in K[[X_1, \dots, X_m]]$ are linearly independent over K , then at least one of their generalized Wronskians is not identically zero ([2, Theorem 3]). In this article we propose an equally short, yet completely different proof of this result. We do so by generalizing it to, F_K , the field of fractions of the formal power series rings over K in countably many indeterminates. More precisely, we prove:

Theorem 1.1. *Let K be a characteristic zero field. A finite family f_1, \dots, f_n of elements of F_K is linearly dependent over K if and only if all generalized Wronskians of the family vanish.*

Our proof of Theorem 1.1 takes advantage of the fact that the converse of (*) holds for differential fields [4, Theorem 6.3.4]:

Theorem 1.2. *Let D be a derivation of a field F . A finite family of f_1, \dots, f_n of elements of F is linearly dependent over the kernel of D if and only if the Wronskian of f_1, \dots, f_n with respect to D is 0.*

We will show in Proposition 3.1 that the vanishing of all generalized Wronskians of a family in F_K implies the vanishing of its Wronskian with respect to the log-derivation which can be defined if K contains a copy of $\mathbb{Q}(\log(k) : k \in \mathbb{N})$. So for those fields K Theorem 1.1 follows from Theorem 1.2 since the kernel of the log-derivation is K (Proposition 2.1). The general case can be argued over an extension of K that contains $\mathbb{Q}(\log(k) : k \in \mathbb{N})$ then use the fact that linear independence is preserved under extension of scalars. We find borrowing a transcendental function, namely the logarithm, to prove a purely algebraic result interesting. In the next section the reader will see that considering the problem in countably many variables allows us to examine it through the lens of arithmetic functions/Dirichlet series. It is only from that point of view that the relevance of logarithm becomes transparent.

2 Ring of Arithmetic Functions

Let K be a field. By a **K -valued arithmetic function** we mean a function from \mathbb{N} to K . With $\alpha \in K$ identified with the function $1 \mapsto \alpha$ and $n \mapsto 0$ for all $n > 1$ the K -valued arithmetic functions form a K -algebra, denoted by A_K , under the operations:

1. $(f + g)(n) := f(n) + g(n)$; and
2. $(f * g)(n) := \sum_{mk=n} f(m)g(k)$.

The following are a few specific \mathbb{Q} -valued arithmetic functions that often appear in this article: let e_n be the function whose value is 1 at n and 0 elsewhere. For each prime p , let v_p be the function that assigns to each n the largest integer

$k \geq 0$ so that p^k divides n . Let Ω be the function $\sum_{p \in \mathbb{P}} v_p$ where p runs through the set of primes \mathbb{P} , thus Ω counts the total number of prime factors (with multiplicity) of its argument.

At first sight the ring A_K does not look like a ring of power series, but it is actually isomorphic, as K -algebra, to $K[[X_p : p \in \mathbb{P}]]$ via

$$f \mapsto \sum_{n \in \mathbb{N}} f(n) \prod_{p \in \mathbb{P}} X_p^{v_p(n)}. \quad (2.1)$$

The algebras $K[[X_p : p \in \mathbb{P}]]$ and $K[[X_i : i \in \mathbb{N}]]$ are certainly isomorphic too, but our results can be stated much more elegantly if A_K is identified with the former algebra. Note also that A_K is isomorphic to the algebra of formal Dirichlet series with coefficients from K via

$$f \mapsto \sum_{n \geq 1} \frac{f(n)}{n^s}. \quad (2.2)$$

We will drop the base field K from the notation if no confusion arises.

For a nonzero $f \in A$, let $\text{ord}(f)$ be the least $n \in \mathbb{N}$ such that $f(n) \neq 0$. We set $\text{ord}(0) = \infty$. Let $\|f\|$ to be $1/\text{ord}(f)$ (with $1/\infty = 0$). Clearly $\|f\| \geq 0$ for any $f \in A$ and the equality holds precisely when $f = 0$. It is also clear that $\|f + g\| \leq \max\{\|f\|, \|g\|\}$. Moreover, one checks that for all $f, g \in A$, $(f * g)(n) = 0$ for all $n < \text{ord}(f) \text{ord}(g)$, and that

$$(f * g)(\text{ord}(f) \text{ord}(g)) = f(\text{ord}(f))g(\text{ord}(g)).$$

Therefore, $\text{ord}(f * g) = \text{ord}(f) \text{ord}(g)$ or equivalently $\|f * g\| = \|f\| \|g\|$. Thus $\|\cdot\|$ is an ultrametric absolute value on A (c.f. [5, Section 1.1]). Consequently, A is an integral domain. Let F denote its field of fractions. Note that $\|\cdot\|$ extends uniquely to an ultrametric absolute value of F by $\|f/g\| := \|f\| / \|g\|$. We further extend $\|\cdot\|$ to a norm on F^m ($m \in \mathbb{N}$) by setting $\|\mathbf{x}\| := \max\{\|x_i\| : 1 \leq i \leq m\}$. A sequence (\mathbf{x}_n) in F^m **converges** to an element $\mathbf{x} \in F^m$, if the sequence $(\|\mathbf{x}_n - \mathbf{x}\|)_{n \in \mathbb{N}}$ converges to 0. Addition and multiplication of F are continuous (i.e. preserving convergent sequences) and so are the coordinate projections. As a result, polynomial functions are continuous. In particular for each n , the determinant function from $F^{n \times n}$ to F is continuous.

A **derivation** of a ring R is a map D from R to itself satisfying $D(a + b) = D(a) + D(b)$ and $D(ab) = aD(b) + bD(a)$ for all $a, b \in R$. We will not distinguish by notation a derivation of an integral domain from its unique extension (by the “quotient rule”) to the field of fractions. There are several “natural” derivations of A (and hence F) coming from the isomorphisms (2.1) and (2.2): the derivation $\partial/\partial X_p$ ($p \in \mathbb{P}$) of $K[[X_p : p \in \mathbb{P}]]$ corresponds to the derivation ∂_p of A given by

$$(\partial_p f)(n) = v_p(np)f(np), \quad (f \in A, n \in \mathbb{N}).$$

We call each ∂_p ($p \in \mathbb{P}$) a **basic derivation** of A . If K contains a copy of the field $\mathbb{Q}(\log(k) : k \in \mathbb{N})$, then the derivation $-d/ds$ of the ring of formal Dirichlet series corresponds to the derivation ∂ of A given by

$$\partial f(n) = \log(n)f(n), \quad (f \in A, n \in \mathbb{N}).$$

We call ∂ the **log-derivation** of A . It is easy to check that $\|f\| = \|\partial f\|$ if and only if $\|f\| < 1$, or equivalently f is not a unit of A . One checks also that the kernel of ∂ in A , i.e. the set $\ker_A \partial := \{f \in A : \partial f = 0\}$, is K . Next we show that extending ∂ to F does not enlarge its kernel.

Proposition 2.1. $\ker_{F_K} \partial = K$.

Proof. The inclusion $K \subseteq \ker_F \partial$ is clear. To establish the reverse inclusion, pick any $f/g \in \ker_F \partial \setminus \{0\}$ then

$$\partial f * g = f * \partial g \quad (2.3)$$

and so $\|\partial f\| \|g\| = \|f\| \|\partial g\|$. If g is invertible in A , then f/g is already in $\ker_A \partial = K$. So suppose g is not a unit, it then follows that $\|g\| = \|\partial g\|$ and hence $\|f\| = \|\partial f\|$ (because $\|g\| \neq 0$). Evaluating both sides of (2.3) at $\text{ord}(f) \text{ord}(g)$ yields $\log(\text{ord}(f)) = \log(\text{ord}(g))$ and hence $\text{ord}(f) = \text{ord}(g)$. Consider the function $h := f - (f(n)/g(n))g$ where $n = \text{ord}(f) = \text{ord}(g)$. Then $\text{ord}(h) > \text{ord}(g)$ but $h/g = f/g - f(n)/g(n) \in \ker_F \partial$. So h must be 0, i.e. $f/g = f(n)/g(n) \in K$, otherwise the same argument with f replaced by h will show that $\text{ord}(h) = \text{ord}(g)$, a contradiction. This completes the proof of the other inclusion. \square

It is probably worth pointing that the validity of the proposition hinges on the fact that \log is 1-to-1. For instance, one checks readily that the kernel of the derivation on A given by $\partial_\Omega f(n) = \Omega(n)f(n)$ is K but for distinct primes p, q , $e_p/e_q \in \ker_F \partial_\Omega \setminus K$.

Every continuous derivation of A can be expressed as a series of basic derivations [11, Theorem 4.3]. The series for the log-derivation is

$$\sum_{p \in \mathbb{P}} \log(p) e_p * \partial_p. \quad (2.4)$$

That means for each $f \in A$, the sequence $(s_N(f))_{N \in \mathbb{N}}$ converges to ∂f where s_N is the partial sums $\sum_{p \leq N} \log(p) e_p * \partial_p$. Next we show that the same relation holds for their extensions to F .

Lemma 2.2. As derivations of F , $\partial = \sum_p \log(p) e_p * \partial_p$.

Proof. We need to show that for any f and $g \neq 0$ in A , $\|s_N(f/g) - \partial(f/g)\| \rightarrow 0$ as $N \rightarrow \infty$. Since s_N is a derivation,

$$\begin{aligned} & \left\| s_N \left(\frac{f}{g} \right) - \partial \left(\frac{f}{g} \right) \right\| \\ &= \left\| \frac{s_N(f) - \partial f}{g} - \left(\frac{f}{g} \right) \frac{s_N(g) - \partial g}{g} \right\| \\ &\leq \max \left\{ \frac{\|s_N(f) - \partial f\|}{\|g\|}, \frac{\|f\| \|s_N g - \partial g\|}{\|g\|^2} \right\}. \end{aligned}$$

The maximum tends to 0 as N tends to ∞ because s_N converges to ∂ on A . \square

Basic derivations commute with each other. In general, two derivations need not commute but their commutator is always a derivation. The next lemma can be viewed as a generalization of this fact to differential operators of the form $\partial_m := \prod_{p \in \mathbb{P}} \partial_p^{v_p(m)}$ ($m \in \mathbb{N}$). Note that ∂_1 is simply the identity map of F .

Lemma 2.3. $[\partial_m, \partial] = \log(m)\partial_m$ on F .

Proof. We prove this by induction on $\Omega(m)$. The case $\Omega(m) = 0$, i.e. $m = 1$ is clear. Assume for some $k \geq 0$, the lemma is true for all n with $\Omega(n) = k$. Pick any $m \in \mathbb{N}$ with $\Omega(m) = k + 1$ then $m = np$ for some prime p and n with $\Omega(n) = k$. So $[\partial_n, \partial] = \log(n)\partial_n$ by the induction hypothesis. One checks directly that the derivations $[\partial_p, \partial]$ and $\log(p)\partial_p$ agree on A and hence on F . Thus, the following computation finishes the proof.

$$\begin{aligned} [\partial_m, \partial] &= \partial_m \partial - \partial \partial_m = \partial_p (\partial_n \partial) - (\partial \partial_p) \partial_n \\ &= \partial_p (\partial \partial_n + \log(n) \partial_n) - (\partial_p \partial - \log(p) \partial_p) \partial_n \\ &= \partial_p \partial \partial_n + \log(n) \partial_p \partial_n - \partial_p \partial \partial_n + \log(p) \partial_p \partial_n = \log(m) \partial_m. \end{aligned}$$

□

3 Wronskians and Linear Dependence

A **generalized Wronskian** of a family $\mathbf{f} = (f_1, \dots, f_n)$ of elements of F is the determinant of a matrix of the form

$$\begin{pmatrix} \partial_{m_1} f_1 & \cdots & \partial_{m_1} f_n \\ \partial_{m_2} f_1 & \cdots & \partial_{m_2} f_n \\ \vdots & \vdots & \vdots \\ \partial_{m_n} f_1 & \cdots & \partial_{m_n} f_n \end{pmatrix} \quad (3.1)$$

where $\Omega(\mathbf{m}) := (\Omega(m_1), \dots, \Omega(m_n))$ is admissible. Here a tuple of non-negative integers is **admissible** if its i -th entry is at most $i - 1$ ($1 \leq i \leq n$). Consequently, $\Omega(m_1)$ must be 0 and so m_1 must be 1 if $\Omega(\mathbf{m})$ is admissible. The concept of generalized Wronskian was introduced by Ostrowski [9] and was used in the proof of the famous Roth Lemma (see, for example, [6, Lemma D.6.1, Proposition D.6.2]).

For notation simplicity, we will drop the fix but otherwise arbitrary family \mathbf{f} from the notation. Also, we will regard a matrix as the tuple of its rows. Hence $(\partial_{\mathbf{m}}) = (\partial_{m_1}, \dots, \partial_{m_n})$ stands for the matrix in (3.1). Similarly, for a tuple $\mathbf{k} = (k_1, \dots, k_n)$ of non-negative integers, $(\partial^{\mathbf{k}}) = (\partial^{k_1}, \dots, \partial^{k_n})$ stands for the matrix $(\partial^{k_i} f_j)$. The determinant of the matrix $(\partial^{i-1} f_j)$ ($1 \leq i, j \leq n$) is the **∂ -Wronskian** of \mathbf{f} , i.e. the Wronskian of \mathbf{f} with respect to the log-derivation.

The next proposition is a key step in our proof of Theorem 1.1. The idea is to “replace” each basic derivation in a generalized Wronskian by the log-derivation successively while keeping the intermediate determinants zero. To avoid this rather simple idea being obscured by the induction argument, we encourage the

reader to work out the case $n = 3$ for himself/herself. In the course of writing out the proof, we find the list-slicing syntax from the programming language Python useful¹: for $\mathbf{k} = (k_1, \dots, k_n)$, let $\mathbf{k}[i:]$ denote the tuple (k_1, \dots, k_i) and $\mathbf{k}[i:]$ denote (k_i, \dots, k_n) . We understood $\mathbf{k}[0]$ and $\mathbf{k}[n+1:]$ to be the null sequence.

Proposition 3.1. *Let K be a field containing a copy of $\mathbb{Q}(\log(k) : k \in \mathbb{N})$ as subfield. If all generalized Wronskians of a family $f_1, \dots, f_n \in F_K$ vanish then so does its ∂ -Wronskian.*

Proof. For $1 \leq i \leq n+1$, let $S_i(\mathbf{k})$ be the statement:

$$\det(\partial^{\mathbf{k}[i-1]}, \partial_{\mathbf{m}}) = 0$$

for any \mathbf{m} with $\Omega(\mathbf{m}) = \mathbf{k}[i:]$. Let S_i be the statement: for all admissible \mathbf{k} , $S_i(\mathbf{k})$. The assumption on the vanishing of all generalized Wronskians means S_1 is true. If we can establish S_{i+1} from S_i , then by induction S_{n+1} is true. Consequently, $\det(\partial^0, \partial^1, \dots, \partial^{n-1}) = 0$; that is the ∂ -Wronskian of \mathbf{f} vanishes since $\mathbf{k} = (0, 1, \dots, n-1)$ is admissible.

To establish S_{i+1} , let $S_{i,j}(\mathbf{k})$ be the statement:

$$\det(\partial^{\mathbf{k}[i-1]}, \partial_\ell \partial^j, \partial_{\mathbf{m}}) = 0$$

for any $\ell \in \mathbb{N}$ with $\Omega(\ell) + j = k_i$ and for any \mathbf{m} with $\Omega(\mathbf{m}) = \mathbf{k}[i+1:]$. For $j \geq 0$, let $S_{i,j}$ assert $S_{i,j}(\mathbf{k})$ for all admissible \mathbf{k} with $k_i \geq j$. Note that $S_{i,0}$ is equivalent to S_i and hence is assumed to be true. Now assume $S_{i,j}$ is true for some $j \geq 0$. We argue that $S_{i,j+1}$ is true. Once this is established then it follows from induction that $S_{i,k_i}(\mathbf{k})$ is true for any admissible \mathbf{k} . Since $S_{i,k_i}(\mathbf{k})$ is equivalent to $S_{i+1}(\mathbf{k})$ and \mathbf{k} is an arbitrary admissible tuple, we establish S_{i+1} .

It remains to prove $S_{i,j+1}$. To do that, pick an arbitrary admissible \mathbf{a} with $a_i \geq j+1$, an arbitrary $b \in \mathbb{N}$ with $\Omega(b) + j+1 = a_i$ and an arbitrary \mathbf{c} with $\Omega(\mathbf{c}) = \mathbf{a}[i+1:]$. We need to show that

$$\det(\partial^{\mathbf{a}[i-1]}, \partial_b \partial^{j+1}, \partial_{\mathbf{c}}) = 0. \quad (3.2)$$

By Lemma 2.3,

$$\begin{aligned} \det(\partial^{\mathbf{a}[i-1]}, \partial_b \partial^{j+1}, \partial_{\mathbf{c}}) &= \det(\dots, \partial \partial_b \partial^j + \log(b) \partial_b \partial^j, \dots) \\ &= \det(\dots, \partial \partial_b \partial^j, \dots) + \log(b) \det(\dots, \partial_b \partial^j, \dots). \end{aligned} \quad (3.3)$$

Note that $a_i - 1 \geq j \geq 0$ and $\mathbf{a}' := (a_1, \dots, a_i - 1, \dots, a_n)$ is still admissible. Because $\Omega(\mathbf{a}[i-1]) = \Omega(\mathbf{a}'[i-1])$, $\Omega(b) + j = a_i - 1$ and $\Omega(\mathbf{c}) = \mathbf{a}[i+1:] = \mathbf{a}'[i+1:]$ it follows from $S_{i,j}(\mathbf{a}')$ that

$$\det(\dots, \partial_b \partial^j, \dots) = 0 \quad (3.4)$$

¹We differ from Python slightly. For instance, in Python indices start at 0 and so $\mathbf{k}[i:]$ means (k_0, \dots, k_{i-1}) instead.

We now claim that

$$\det(\cdots, \partial \partial_b \partial^j, \cdots) = 0 \quad (3.5)$$

as well. This is because for each prime p , $\Omega(pb) + j = \Omega(b) + j + 1 = a_i$ and so by $S_{i,j}(\mathbf{a})$

$$\det(\partial^{\mathbf{a}^{[i-1]}}, \partial_p \partial_b \partial^j, \partial_c) = \det(\partial^{\mathbf{a}^{[i-1]}}, \partial_{pb} \partial^j, \partial_c) = 0. \quad (3.6)$$

Now multiply Equation (3.6) by $\log(p)e_p$ then sum through the primes, we conclude from Lemma 2.2 and the continuity of determinant that Equation (3.5) holds. Finally, Equation (3.3), (3.4) and (3.5) together imply Equation (3.2). Since \mathbf{a} is an arbitrary admissible tuple with $a_i \geq j + 1$, we establish $S_{i,j+1}$ and complete the proof. \square

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Suppose $\sum_i c_i f_i = 0$, where $c_1, \dots, c_n \in K$ not all zero, witnessing the linear dependence of f_1, \dots, f_n over K . Since $\ker \partial_m \supseteq K$ for each $m > 1$ so

$$0 = \partial_m \sum c_i f_i = c_1 \partial_m f_1 + \cdots + c_n \partial_m f_n.$$

This equation holds for $m = 1$ as well since ∂_1 is the identity operator. Thus, for any $\mathbf{m} \in \mathbb{N}^n$, $\mathbf{c} = (c_1, \dots, c_n)$ is a non-trivial solution to the linear system $(\partial_{\mathbf{m}} \mathbf{f}) \mathbf{x} = \mathbf{0}$. Hence every generalized Wronskian of f_1, \dots, f_n vanishes.

To prove the other implication, let L be a field that extends K and contains a copy of $\mathbb{Q}(\log(k) : k \in \mathbb{N})$. The existence of such L is guaranteed by [3, Ch 5, Prop 4]. For any $f_1, \dots, f_n \in F_K$, if their generalized Wronskians all vanish then, by Proposition 3.1, so does their ∂ -Wronskian. It then follows from Theorem 1.2 that f_1, \dots, f_n are linearly dependent over the kernel of ∂ in F_L which is L by Proposition 2.1. But then they must also be linearly dependent over K . An elementary way of seeing this is as follows: view each f_i as a row vector $(f_{ij} : j \in \mathbb{N})$ where $f_{ij} = f_i(j)$. If they were linearly independent over K , then by Gaussian elimination, there exists a sequence of elementary row operations over K that turns f_1, \dots, f_n into a family of non-zero row vectors with straightly increasing orders (viewed as arithmetic functions). Since L extends K , the same sequence of operations can be carried out over L showing that f_1, \dots, f_n are linearly independent over L , a contradiction. \square

If we view $f \in K[[X_1, \dots, X_m]]$ as an element of $K[[X_i : i \in \mathbb{N}]]$, it is clear that $(\partial/\partial X_j)f = 0$ for all $j > m$. Hence, if the generalized Wronskians of a family in $K[[X_1, \dots, X_m]]$ all vanish, then the same is true when it is regarded as a family in $K[[X_i : i \in \mathbb{N}]]$. Thus Theorem 1.1 generalizes Theorem 3 of [2]. At the same time, Theorem 1.1 also generalizes Theorem 2.1 of [13] since the ring of germs of analytic functions at the origin of K^m (where $K = \mathbb{R}$ or \mathbb{C}) and hence its fraction field embeds into F_K . In the same article, the author also identified the smallest collection of generalized Wronskians whose vanishing implies linear dependence of the family over the constants [13, Theorem 3.1, 3.4].

This collection, in our set up, are those generalized Wronskians indexed by tuples with the property that a divisor of an entry is also an entry; for instance, in the case $n = 3$, the non-trivial assumptions are the vanishing of $\det(\text{id}, \partial_p, \partial_{p^2})$ and $\det(\text{id}, \partial_p, \partial_q)$ for all primes p and q . One can establish these results for the field F_K by mimicking the existing proofs. However, we are unable to do so using arguments similar to those proposed here. Finally, we would like to mention that this article was inspired by our study of another type dependence—algebraic dependence—of arithmetic functions. The interplay between formal power series, arithmetic functions and formal Dirichlet series is also proved to be fruitful in that context [7, 10, 12].

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